Linear Algebra

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1. **Vector Spaces**

1.A $\mathbb{R}^n$ and $\mathbb{C}^n$

Complex Numbers

You should already be familiar with basic properties of the set $\mathbb{R}$ of real numbers.

Complex numbers were invented so that we can take square roots of negative numbers.

The idea is to assume we have a square root of ??-1, denoted i, that obeys the usual rules of arithmetic.

Here are the formal definitions:

Definition: complex numbers

- A **complex number** is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- **Addition and multiplication** on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

here $a, b, c, d \in \mathbb{R}$.

If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a. Thus we can think of $\mathbb{R}$ as a subset of $\mathbb{C}$.

We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i.

Using multiplication as defined above, you can verify that $i^2 = -1$.

Properties of complex arithmetic

commutativity

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \text{ for all } \alpha, \beta \in \mathbb{C}$$

associativity

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\lambda = \alpha(\beta\lambda) \text{ for all } \alpha, \beta, \lambda \in \mathbb{C}$$

identities

$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda \text{ for all } \lambda \in \mathbb{C}$$

additive inverse

$$\text{for every } \alpha \in \mathbb{C}, \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha + \beta = 0$$

multiplicative inverse

$$\text{for every } \alpha \in \mathbb{C} \text{ with } \alpha \neq 0, \text{ there exists a unique } \beta \in \mathbb{C}$$

$$\text{such that } \alpha\beta = 1$$

distributive property

$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \text{ for all } \alpha, \beta, \lambda \in \mathbb{C}$$

The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication.

Definition: $-\alpha$, subtraction, $1/\alpha$, division

Let $\alpha, \beta \in \mathbb{C}$,

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0$$

- Subtraction on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha)$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1$$

- Division on $\mathbb{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha)$$

In order to make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation:

Throughout these notes, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Thus if we prove a theorem involving $\mathbb{F}$, we will know that it holds when $\mathbb{F}$ is replaced with $\mathbb{R}$ and when $\mathbb{F}$ is replaced with $\mathbb{C}$.

Elements of $\mathbb{F}$ are called **scalars**. The word "scalar", a fancy word for "number", is often used when we want to emphasize that an object is a number, as opposed to a vector (vectors to be defined).

For $\alpha \in \mathbb{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}$$

Clearly, $(\alpha^m))^n = \alpha^{mn}$ and $(\alpha\beta)^m = \alpha^m \beta^m$ for all $\alpha, \beta \in \mathbb{F}$ and all positive integeres m, n.